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In Search of the "Perfect" Straight Line and Constant Velocity Too

Design Charts for Optimum Straight-Line Crank-Rocker Fourbar Linkages With Near Constant Velocity

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Abstract

Cam-follower systems can easily be designed to accomplish straight-line motion with constant velocity but are typically massive, expensive and can have dynamic problems at high speeds. The fourbar linkage is a potentially simpler and cheaper alternative. Though it is not possible to achieve either exact straight line motion or exact constant velocity with a fourbar linkage, with the right linkage geometry, these position and velocity constraints can be very closely approximated over significant portions of the cycle. This paper describes a set of Grashof crank-rocker fourbar linkage geometries which will deliver minimum structural error in either position or velocity deviation over various portions of the crank cycle when the crank is driven at a constant angular velocity. Design charts are provided to allow selection of the proper linkage geometry for the desired situation. Deviations of less than 1% in position accuracy and less than a few percent of constant velocity are possible over up to 50% of the cycle. The fourbar linkage can be balanced by conventional techniques to reduce or eliminate its shaking forces and moments and will typically allow high speed operation with fewer dynamic problems than cam-follower systems.

Introduction

Straight-line linkages have been known and used since the time of James Watt in the 18th century. Many kinematicians such as Watt, Chebychev, Peaucellier, Kempe, Evans, Hoekens, and others have developed or discovered either approximate or exact straight line linkages, and their names are associated with those devices to this day. It is well known that a fourbar pin-joined linkage is capable of generating only approximate straight lines. Their sixth-order coupler curves can intersect a true straight line in only six places and will deviate to some degree from the straight path at all other points. At least six links and seven pin joints are needed to generate an exact straight line with a pure linkage, i.e., a Watt's or Stephenson's sixbar. A geared fivebar mechanism, with a gear ratio of -1 and a phase angle of π, will generate an exact straight line but it is merely a transformed Watt's sixbar obtained by replacing one binary link with a higher joint in the form of a gear pair. The Peaucellier linkage uses eight bars and six pins to generate an exact straight line and provides the bonus of generating true circle arcs of large radius in response to small adjustments of its ground link length from the straight-line position.

The Need

Given all of the above well-known information, and the fact that an exact straight line can be generated with six or more links, why does the quest for "better fourbar straight-line linkages" or improved means for their general design continue? One reason is the desire for simplicity in machine design. The pin-jointed fourbar is the simplest possible one-degree-of-freedom mechanism. Another reason is that a very good approximation to a true straight line can be obtained with just four bars, and this is often "good enough" for the needs of the machine being designed. Manufacturing tolerances will, after all, cause any mechanism's performance to be less than ideal. As the number of links and joints increases, the probability that an exact straight line mechanism will deliver its theoretical performance in practice is obviously reduced.

Finally, there is a real need for straight-line motions in machinery of all kinds, especially in automated production machinery. Many consumer products such as cameras, film, toiletries, razors, bottles, etc. are manufactured, decorated, or assembled on sophisticated and complicated machines that contain a myriad of linkages and cam-follower systems. Traditionally, most of this kind of production equipment has been of the intermittent-motion variety. This means that the product is carried through the machine on a linear or rotary conveyor that stops for any operation to be done on the product, and then indexes the product to the next work station where it again stops for another operation to be performed. The forces and power required to accelerate and decelerate the large mass of the conveyor (which is independent of, and typically larger than, the mass of the product) severely limit the speeds at which these machines can be run. Economic considerations continually demand higher production rates, requiring higher speeds or additional, expensive machines. This economic pressure has caused many manufacturers to redesign their assembly equipment for continuous conveyor motion. When the product is in continuous motion in a straight line and at constant velocity, every workhead that operates on the product must be articulated to chase the product and match both its straight-line path and its constant velocity while performing the task. These factors have increased the need for straight-line mechanisms, including ones capable of near-constant velocity over the straight-line path.

Watt originally devised his straight line linkage in order to guide the piston of his steam engine at a time when metal-cutting machinery that could create a long, straight guideway did not yet exist. Since we no longer suffer that limitation, a (near) perfect straight line motion is easily obtained with a fourbar slider-crank mechanism. Ball-bushings and hardened ways are available commercially at moderate cost and make this a reasonable, low-friction solution to the straight-line path guidance problem. But, the cost of a properly guided slider-crank mechanism is still greater than that of a pin-jointed fourbar linkage. Moreover, a crank-slider-block has a velocity profile that is nearly sinusoidal (with some harmonic content) and is far from constant velocity over all of its motion. This problem has been addressed by Hain (1967), who used a dyad to drive the input crank of the slider-crank linkage making it a Watt's sixbar whose first, drag-link stage effectively modified the harmonic motion of the slider to have a significant constant velocity portion. Shoen (1962) describes a practical application of this technique. Hodges and Pisano (1991) describe the synthesis of sixbar linkages for straight line, constant velocity, scanning devices. Nevertheless, all these approaches result in a more complicated and expensive solution than can be attained with just four pin-jointed links. This paper will show that we can achieve both near-exact straightness of path and near-constant velocity in the direction of path motion in the same crank-rocker, pin-jointed, fourbar linkage.

Previous Work

In recent decades, much effort has been devoted by many researchers toward successfully establishing a solid understanding of the kinematic theory affecting the design of straight line mechanisms, particularly the fourbar linkage, based on work done or reported over a century ago by Kempe (1877), Roberts (1876), Cayley (1876), Burmester (1888), and others. In the last several decades, Freudenstein (1965), Kearney (1954), Hall (1958), Wolford (1960), Tesar (Tesar and Vidosic 1967; Tesar et al. 1967; Tesar and Watts 1966; Tesar and Wolford 1960; Tesar and Wolford 1962), Vidosic (1967), Pennock (1991), and many others have investigated the use of higher path curvature analysis, the inflection circle, Burmester's points, Ball's point, instantaneous invariants, and combinations of these and other methods for the purpose of defining and designing fourbar linkages that will give approximate straight line motions. These theories will not be discussed here. The reader is directed to the references for complete information.

A paper on this subject by Bunduwongse and Ting (1991), uses a double-fifth-order Ball's Burmester point at the inflection pole to define a unique linkage that will generate a highly accurate straight line (effectively passing in a straight path through five infinitesimally separated points in the plane). This linkage is shown in Figure 1 which is reproduced from their paper. The fifth-order curvature in this case predicts a path that should be maximally "flat" in the region of the Ball's Burmester point.

Discussion

Interestingly, the linkage in Figure 1 appears, at a glance, to be the well-known Chebychev symmetrical straight-line linkage, suggesting that Chebychev may have gotten it exactly right so long ago (and without a computer!). But, the link ratios defined by Bunduwongse and Ting for this optimal straight-line generator are not the same as those defined by the famous Russian kinematician. Taking the shortest link (coupler) as 1 unit in both cases and proceeding around the loop in either direction, the Chebychev linkage has the normalized dimensions: 1, 2.5, 2, 2.5, with the cou-

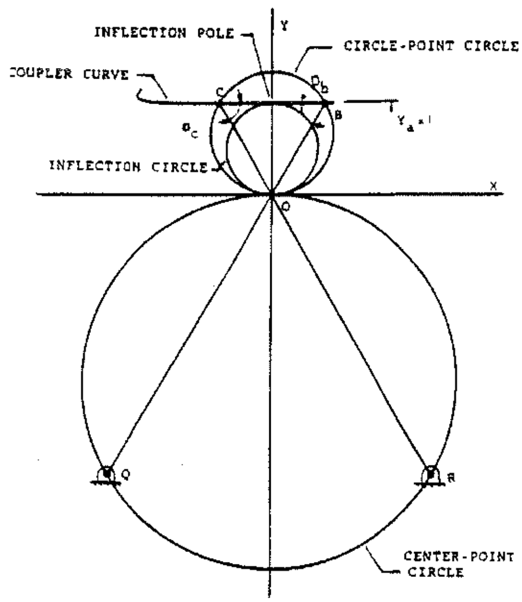

Figure 1 Optimum linkage configuration for double 5th order Ball's Burmester point at inflection pole from Bunduwongse and Ting (1991).

pler point midway along the line of the coupler.* The Bunduwongse-Ting (*BT*) linkage has the normalized dimensions: 1, 4, 3, 4 with the coupler point also midway along the line of the coupler. The two linkages appear to be similar and in fact generate a coupler curve of essentially the same overall shape.

The Chebychev linkage and the *BT* linkage are both Grashof double-rockers in which the coupler can rotate fully, but they will reach toggle positions if driven from either of their rockers. To use them in a practical application, it is necessary either to add a driver dyad, making a Watt's sixbar, or to drive via chain or belt from a ground pivot to a moving pivot, thus imparting rotary motion to the shortest link. Either of these approaches adds undesired complexity and cost.

Another possible approach is to determine the two fourbar cognate linkages which generate the same coupler path and use one of them instead. A discussion of cognates and a method to find the cognates of a fourbar linkage whose

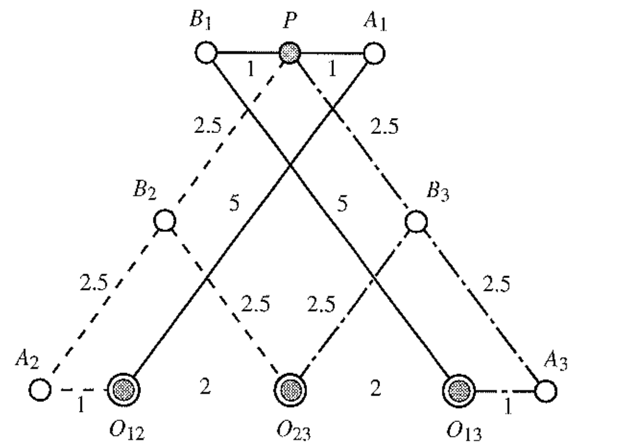

Figure 2 Roberts diagram showing the Chebychev linkage and its two cognates, all of which generate the same coupler curve at point P

coupler point lies on the line of centers of the coupler can be found in Hartenberg (1958; 1959). A Roberts diagram depicting the Chebychev linkage and its two cognates is shown in Figure 2 in which O_{12}, A_1, B_1, O_{13} is the Chebychev linkage, O_{12}, A_2, B_2, O_{23} is the first cognate, and O_{13}, A_3, B_3, O_{23} is the second cognate. All three share the common coupler point *P* and generate the identical coupler curve shown in Figure 3. Interestingly, the Chebychev linkage's two fourbar cognates are "mirror twins." Both cognates are the somewhat less well-known Hoeken

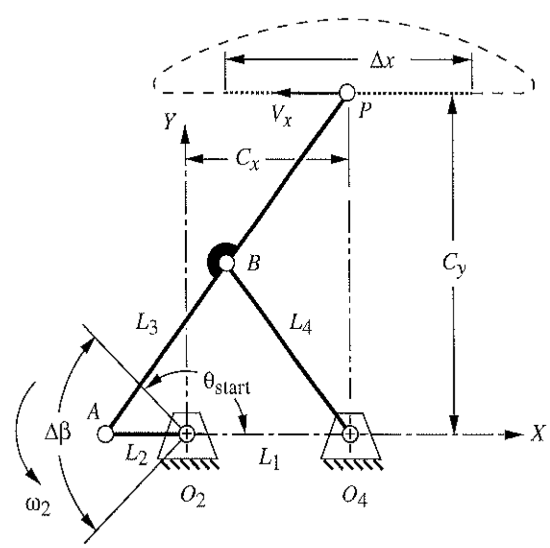

Figure 3 Hoeken crank-rocker straight-line linkage and its coupler curve shared with its Chebychev linkage cognate.

* The link ratios of the Chebychev straight-line linkage have been reported differently by various authors. The ratios used here are those first reported (in English) by Kempe (1877). Kennedy (1893) describes the same linkage, reportedly "as Chebychev demonstrated it at the Vienna Exhibition of 1893) as having the ratios 1, 3.25, 2.5, 3.25 which are closer to, but not the same as, the *BT* ratios for optimum straightness.

straight-line linkage (Hoeken 1926).* Hoeken's linkage has the crank-normalized dimensions 1, 2.5, 2.5, 2 with the coupler point at 5 crank units from the crank-coupler pivot on the coupler line extended as shown in Figures 2 and 3. This is a Grashof crank-rocker linkage and also has the useful feature of possessing nearly constant velocity along a large portion of its straight-line motion when its crank is driven at constant angular velocity (Norton 1992). This linkage will give a velocity of the coupler point that deviates less than 2% in magnitude along the approximate straight-line portion of its coupler curve over 120 degrees of crank rotation when the crank is driven at constant angular velocity. Because of these two important and practical features, in this paper we will analyze the differences between the original Hoeken linkage and a "Hoeken's family" cognate of the *BT* linkage shown in Figure 1.

The question that came to mind when making the connection between the Chebychev/Hoeken (*CH*) cognate linkages and their geometrically similar *BT* cousins was, "If the *BT* linkage is optimal for straightness of path from a kinematic-theory standpoint (and is thus presumably better than the Chebychev linkage in that regard), is it also optimal for near-constant velocity (and thus better than the Hoeken in that regard)?" To find out, an analysis of the structural errors for both position and velocity over various crank-angle ranges $\Delta\beta$ of straight line coupler-path motion was undertaken for both the *CH* and *BT* geometries and also for other similar linkages over a range of variation of link-ratios. This paper reports on the results of these analyses and defines Hoeken-type crank-rocker linkage geometries that are optimal for accuracy of path straightness and near-constant velocity.

Linkage Geometry

We chose to work only with the Hoeken cognates for the reasons described above. Denoting the link lengths as L_1, L_2, L_3, L_4 for ground, crank, coupler, and rocker respectively (see Figure 3), and taking the crank length as unity for a reference, this symmetrical linkage can be defined by only two link ratios, L_1 / L_2 and L_3 / L_2 (or L_4 / L_2), since $L_3 = L_4$.

Table 1 Link Ratios for Hoeken and BT Linkages

	L1 / L2	L3 / L2	L4 / L2	P / L2	x
Hoeken	2.0	2.5	2.5	5.0 @ 0°	0
BT cognate	3.0	4.0	4.0	8.0 @ 0°	4

These link ratios are shown in Table 1 along with the location of the coupler point in polar coordinates referenced to the line of centers of the coupler. If we take the Hoeken's linkage parameters as representing one location in a linkage "design space" and the linkage parameters of the "Hoeken's" cognate of the *BT* linkage as a second location, we can map this design space to define a spectrum of linkages that can be expected to possess similar properties in respect to their path straightness and velocity characteristics. Figure 4 shows these link ratios plotted against an arbitrary link ratio parameter x. The Hoeken link ratios are arbitrarily assigned to $x = 0$, and the *BT* link ratios to $x = 4$ as shown in Table 1. The equations of the lines connecting the two sets of link ratios are then:

$$\frac{L_1}{L_2} = \frac{1}{4}x + 2 \tag{1}$$

$$\frac{L_3}{L_2} = \frac{L_4}{L_2} = \frac{3}{8}x + 2.5 \tag{2}$$

This link ratio parameter x provides a single variable that defines a linkage in the design space for this class of fourbar mechanism. (Note that the location of the coupler point

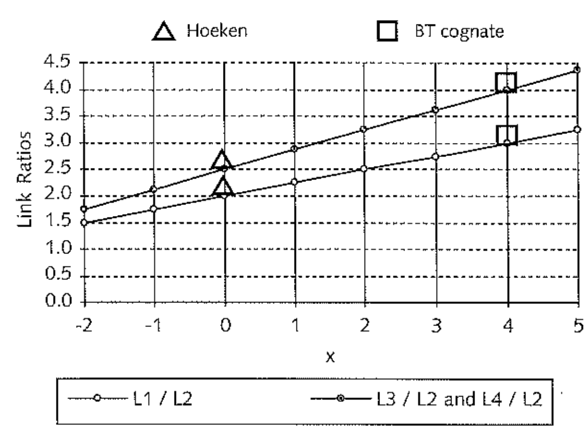

Figure 4 Variation of link ratios with parameter x over the design space

* Hain (1967) cites the Hoekens reference (in German) for this linkage, Nolle (1975) shows the same (Hoekens) mechanism but refers to it as a Chebychev crank rocker without noting its cognate relationship to the Chebychev double rocker also shown. It is certainly conceivable that Chebychev, as one of the creators of the theorem of cognate linkages, would have discovered the "Hoekens" cognate of his own double rocker. However, this author has been unable to find any mention of its genesis in the English literature other than the ones cited above.

Table 2 Required Link Ratios for the Smallest Attainable Errors in Position and Velocity for Various Crank Angle Ranges of a Hoeken–type Straight Line Linkage

Range of Motion			Optimized for Straightness						Optimized for Constant Velocity					
Angle range Δβ(deg)	start angle θ1(deg)	% of crank cycle	smallest y deviation %	V deviation %	at x	L1 / L2	L3 / L2 and L4 / L2	Δx / L2	smallest V deviation %	y deviation %	at x	L1 / L2	L3 / L2 and L4 / L2	Δx / L2
20	170	5.6%	0.00001%	0.38%	3.90	2.975	3.963	0.601	0.006%	0.137%	0.30	2.075	2.613	0.480
40	160	11.1%	0.00004%	1.53%	3.80	2.950	3.925	1.193	0.038%	0.274%	0.20	2.050	2.575	0.950
60	150	16.7%	0.00027%	3.48%	3.60	2.900	3.850	1.763	0.106%	0.387%	0.10	2.025	2.538	1.411
80	140	22.2%	0.001%	6.27%	3.30	2.825	3.738	2.299	0.340%	0.503%	–0.10	1.975	2.463	1.845
100	130	27.8%	0.004%	9.90%	2.90	2.725	3.588	2.790	0.910%	0.640%	–0.40	1.900	2.350	2.237
120	120	33.3%	0.010%	14.68%	2.50	2.625	3.438	3.238	1.885%	0.752%	–0.70	1.825	2.238	2.600
140	110	38.9%	0.023%	20.48%	2.00	2.500	3.250	3.623	3.327%	0.888%	–1.00	1.750	2.125	2.932
160	100	44.4%	0.047%	27.15%	1.40	2.350	3.025	3.933	5.878%	1.067%	–1.30	1.675	2.013	3.232
180	90	50.0%	0.096%	35.31%	0.80	2.200	2.800	4.181	9.299%	1.446%	–1.70	1.575	1.863	3.456

is always at twice the length of the coupler on its line of centers extended.) The performance of various linkages can then be investigated in respect to this link ratio parameter x and compared.

Linkage Analysis

A computer program* was written to calculate the Cartesian coordinates of the coupler curve and the velocity of the coupler point for any value of crank angle using a constant crank angular velocity of unity over a linkage ratio range within the design space defined by $-2 < x < 5$. The equations for this kinematic analysis can be found in any introductory kinematic text such as Norton (1992). As shown in Figure 3, the straight-line portion Δx of the Hoeken's linkage coupler curve is parallel to the ground link's line of centers and is symmetrical about a vertical line through the coupler point's position at crank angle $\theta_2 = 180°$ and the fixed pivot O_4. For a global XY coordinate system taken at the crank center with the X-axis oriented along the ground link, the straight line portion of the coupler curve for any linkage in this design space will vary slightly around a particular nominal y value Cy and its velocity of interest along the straight line will be the x component of the coupler point velocity, V_x.

The structural error in position over the straight line portion of the coupler curve was calculated for a set of crank angle ranges Δβ from 20° to 180° centered on the midpoint of the straight portion of the symmetrical coupler curve as shown in Table 2. This position error ε_S was calculated as the maximum y-direction deviation of the straight line over the coupler curve portion defined by the particular crank angle range Δβ, normalized to the length of the straight line segment associated with Δβ in each case.

$$\varepsilon_S = \frac{MAX_{i=1}^{n}\left(C_{y_i}\right) - MIN_{i=1}^{n}\left(C_{y_i}\right)}{\Delta x} \tag{3}$$

The structural error in the velocity ε_V was calculated as the maximum deviation of V_x from the average V_x value of the coupler point over the coupler curve portion defined by the particular β normalized to the average velocity over the length of the straight line segment associated with Δβ.

$$\varepsilon_V = \frac{MAX_{i=1}^{n}\left(V_{x_i}\right) - MIN_{i=1}^{n}\left(V_{x_i}\right)}{\overline{V}_x} \tag{4}$$

The value of the link parameter ratio x was varied from –2 to +5 in increments of 0.1 and the structural errors computed separately for each of nine crank angle ranges Δβ as shown in Table 2. Note that $x = 0$ corresponds to the link ratios of the classic Hoeken linkage and $x = 4$ corresponds to the link ratios suggested by Bunduwongse and Ting (1991). The values of x that provided the minimum structural errors in both position and velocity for each crank angle range Δβ were recorded and these error function data were transferred to a plotting package.

Results

Figure 5 and Table 2 show the relationship between the % error in position (straightness) and the link ratio x for a range of duration angles Δβ. As expected, the BT linkage ($x = 4$) provides the smallest error in position for short portions of the straight line centered around the Ball's

* Available from the author

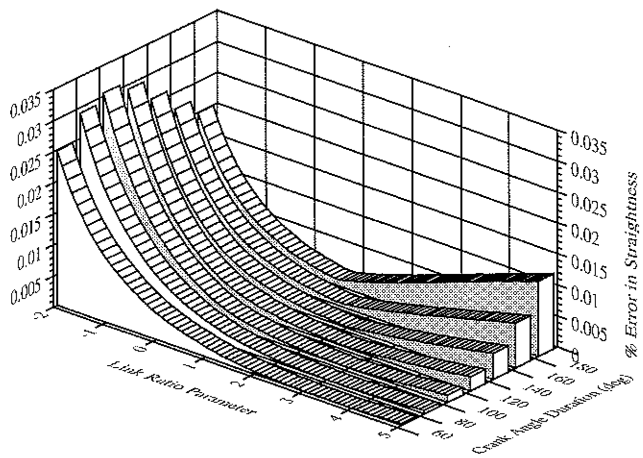

Figure 5 Variation of position error with link ratio parameter x and crank duration of straight line

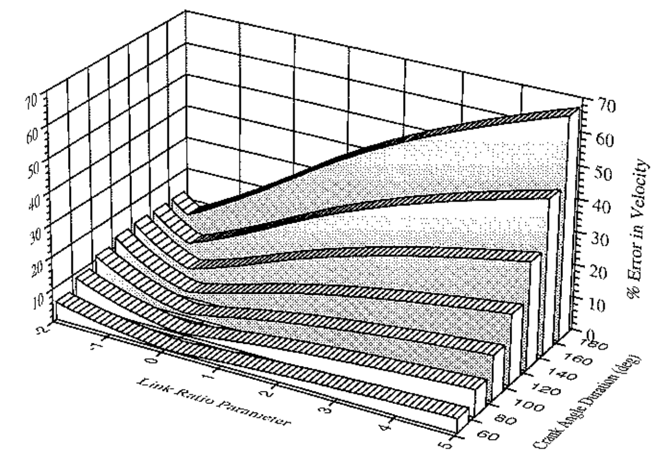

Figure 7 Variation of velocity error with link ratio parameter x and crank duration of straight line

Burmester point as evidenced by the negligible error over $\Delta\beta = 20°$. However, as the length of the straight line portion of the coupler curve used is increased, the optimum value of the link ratio parameter x shifts downward toward a value close to 1 for a crank angle duration of about 180° as shown in Figure 6. At $\Delta\beta = 180°$, the *BT* linkage ($x = 4$) has a maximum deviation from straightness of about 1%, which is still respectably small, but is 10 times the smallest deviation attainable (for $\Delta\beta = 180°$) of 0.1% when $x = 0.80$

Figure 7 and Table 2 show the corresponding relationship between the % error in constant velocity and the link ratio x for the same range of duration angles $\Delta\beta$ as in Figure 5. Note that the smallest deviation in velocity (0.006%) for small $\Delta\beta$ (i.e., 20°) is obtained with a linkage of $x = 0.3$, while the *BT* linkage geometry ($x = 4$) has greater deviation in velocity at small $\Delta\beta$. The velocity deviation of the *BT* linkage increases significantly as more crank duration $\Delta\beta$ of the straight line portion of the coupler curve is used. The optimum value of the link ratio parameter x moves from zero toward –2 as $\Delta\beta$ approaches 180° as can be seen in Figure 8, which also shows that the original Hoeken linkage ($x = 0$) is optimal for velocity accuracy when $\Delta\beta \cong 70°$.

Table 2 provides detailed data on the optimum values of x and their corresponding link ratios that give the smallest possible structural error in either position or velocity over values of $\Delta\beta$ from 20 to 180°. This table and accompanying figures can be used as design charts along with equations 1 and 2 to define a suitable linkage geometry to obtain acceptable deviations in position and velocity over any crank angle duration desired.

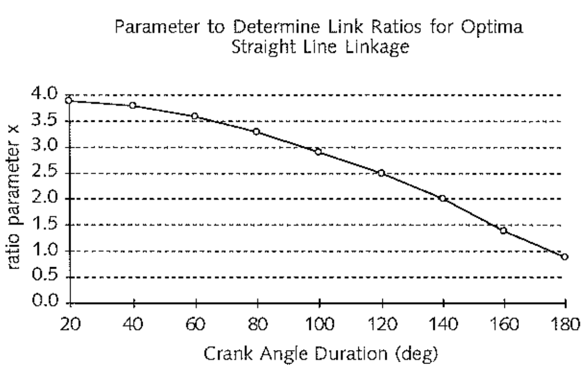

Figure 6 Link ratio parameter x for optimal straight line linkage versus crank angle duration

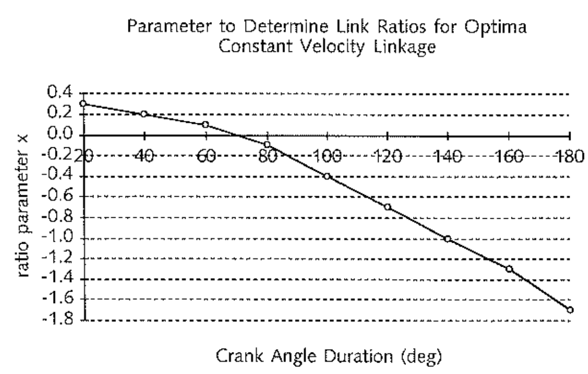

Figure 8 Link ratio parameter x for optimal constant velocity linkage versus crank angle duration

Summary and Conclusions

Many practical machine design problems require both reasonably accurate straight-line motion and near-constant velocity. While some applications require "exactness" in these parameters, most do not, and these latter cases can be satisfied by pure-linkage solutions that provide good approximations. The "exact" requirements are usually met with a cam-follower design at much greater expense than a fourbar linkage. This study shows that very small structural errors in either position deviation from a straight path or velocity deviation from a desired constant value can be obtained from a Hoeken-type crank-rocker fourbar linkage. A reasonable compromise between the accuracy of both position and velocity (< 1% error) can be obtained in the same linkage for moderate values of $\Delta\beta$ up to about 100°. Also, a linkage with $x = -0.7$ will provide a maximum deviation in coupler path straightness of less than 1% and a maximum deviation from constant velocity of less than 2% over 1/3 of the total cycle as can be seen in the sixth row of Table 2.

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